Solutions of the CW Equations

The CW Equations have solutions which can be obtained analytically under certain special cases. This section addresses two such circumstances and illustrates the solution under a generalized acceleration. These derivations shall use the Cartesian form of the CW Equations.

A. No Acceleration

The no-acceleration special case of the CW Equations is given by

$$0 = (d^2x/dt^2) - 2\omega\,(dz/dt)$$
$$0 = (d^2y/dt^2) + \omega^2\, y$$
$$0 = (d^2z/dt^2) + 2\omega\,(dx/dt) - 3\omega^2\, z$$

One can choose among several different methods of solving the set of differential equations, above. One that I find particular straight-forward uses Laplace Transformations to transform a set of differential equations into a set of algebraic equations. The method of Laplace Transformations makes use of the following transformations, summarized below:

$$L\{\, f(t)\,\} = F(s)$$
$$L\{\, df(t)/dt\,\} = s\,F(s) - f\,(0+)$$
$$L\{\, d^2f(t)/dt^2\,\} = s^2\,F(s) - s\,f\,(0+) - f\,'(0+)$$

where f (0+) and f ’(0+) denote the original function and its derivative evaluated as the argument approaches zero from the right (i.e., the initial values of the original function and its derivative).

Taking the Laplace Transform of the no-acceleration special case of the CW Equations yields the following:

$$0 = [\, s^2\,X(s) - s\,x(0+) - x'(0+)\,] - 2\omega\,[\, s\,Z(s) - z(0+)\,]$$
$$0 = [\, s^2\,Y(s) - s\,y(0+) - y'(0+)\,] - \omega^2\,Y(s)$$
$$0 = [\, s^2\,Z(s) - s\,z(0+) - z'(0+)\,] + 2\omega\,[\, s\,X(s) - x(0+)\,] - 3\omega^2\,Z(s)$$

Basically, we’ve transformed the set of differential equations expressed as functions of {x(t), y(t), z(t)} into a set of algebraic equations expressed as functions of {X(s), Y(s), Z(s)}. The elements {x(0+), y(0+), z(0+), x’(0+), y’(0+), z’(0+)} are nothing more than constants.

The second equation is a function of Y(s) only, and thus can be solved separately. The other two equations are a pair of simultaneous equations in X(s) and Z(s) to be solved by whatever means available (left as an exercise to the reader). The solutions to the above system of equations are as follows:

$$\begin{aligned} X(s) = {} & x'(0+)/s^2 - 2\omega z(0+)/s^2 + x(0+)/s + 2\omega z(0+)/(s^2+\omega^2) + \\ & 2\omega z'(0+)/[s(s^2+\omega^2)] + 8\omega^3 z(0+)/[s^2(s^2+\omega^2)] - \\ & 4\omega^2 x'(0+)/[s^2(s^2+\omega^2)] \end{aligned}$$

$$Y(s) = y'(0+)/(s^2+\omega^2) + sy(0+)/(s^2+\omega^2)$$

$$\begin{aligned} Z(s) = {} & z'(0+)/(s^2+\omega^2) + sz(0+)/(s^2+\omega^2) + 4\omega^2 z(0+)/[s(s^2+\omega^2)] - \\ & 2\omega x'(0+)/[s(s^2+\omega^2)] \end{aligned}$$

The next step is to take the inverse Laplace Transform of each equation term-by-term and simplify. The following inverse Laplace Transforms are required:

$$\begin{aligned} L^{-1}\{ 1/s \} &= 1 \\ L^{-1}\{ 1/s^2 \} &= t \\ L^{-1}\{ 1/(s^2+\omega^2) \} &= (1/\omega) \sin \omega t \\ L^{-1}\{ s/(s^2+\omega^2) \} &= \cos \omega t \\ L^{-1}\{ 1/[s(s^2+\omega^2)] \} &= (1/\omega^2)(1 - \cos \omega t) \\ L^{-1}\{ 1/[s^2(s^2+\omega^2)] \} &= (1/\omega^3)(\omega t - \sin \omega t) \end{aligned}$$

Using the above inverse Laplace Transforms and simplifying, we get

$$\begin{aligned} x(t) = {} & [\, 2z'(0+)/\omega + x(0+) \,] + [\, 6\omega z(0+) - 3x'(0+) \,]\, t - \\ & [\, 2z'(0+)/\omega \,] \cos \omega t + [\, 4x'(0+)/\omega - 6z(0+) \,] \sin \omega t \end{aligned}$$

$$y(t) = y(0+) \cos \omega t + [\, y'(0+)/\omega \,] \sin \omega t$$

$$\begin{aligned} z(t) = {} & [\, 4z(0+) - 2x'(0+)/\omega \,] + \\ & [\, 2x'(0+)/\omega - 3z(0+) \,] \cos \omega t + [\, z'(0+)/\omega \,] \sin \omega t \end{aligned}$$

Taking the derivative of the above yields the velocity equations:

$$\begin{aligned} x'(t) = {} & [\, 6\omega z(0+) - 3x'(0+) \,] + \\ & [\, 2z'(0+) \,] \sin \omega t + [\, 4x'(0+) - 6\omega z(0+) \,] \cos \omega t \end{aligned}$$

$$y'(t) = -\omega y(0+) \sin \omega t + [\, y'(0+) \,] \cos \omega t$$

$$z'(t) = [\, 3\omega z(0+) - 2x'(0+) \,] \sin \omega t + [\, z'(0+) \,] \cos \omega t$$

These are the solutions to the no-acceleration case of the CW Equations, as presented at the beginning of this report.

B. Constant Acceleration

The constant-acceleration special case of the CW Equations is given by

$$a_x = (d^2x/dt^2) - 2\omega\,(dz/dt)$$
$$a_y = (d^2y/dt^2) + \omega^2\,y$$
$$a_z = (d^2z/dt^2) + 2\omega\,(dx/dt) - 3\omega^2\,z$$

where $\{a_x, a_y, a_z\}$ are constants. As before, we shall use Laplace Transforms to solve this system of differential equations.

Taking the Laplace Transform of the above yields the following:

$$a_x/s = [\,s^2\,X_a(s) - s\,x(0+) - x'(0+)\,] - 2\omega\,[\,s\,Z_a(s) - z(0+)\,]$$
$$a_y/s = [\,s^2\,Y_a(s) - s\,y(0+) - y'(0+)\,] - \omega^2\,Y_a(s)$$
$$a_z/s = [\,s^2\,Z_a(s) - s\,z(0+) - z'(0+)\,] + 2\omega\,[\,s\,X_a(s) - x(0+)\,] - 3\omega^2\,Z_a(s)$$

Here, we use the subscript “a” on the Laplace Transform functions to distinguish them from the no-acceleration case. The reason for this will become apparent in a moment.

As in the no-acceleration case, we are left with a set of simultaneous equations to solve. In doing so, note that the solution is nearly the same as the no-acceleration case (an exercise left to the reader, again) with a few added terms:

$$X_a(s) = X(s) + a_x/s^3 + 2\omega\,a_z/\,[s^2(s^2+\omega^2)] - 4\omega^2 a_x/\,[s^3(s^2+\omega^2)]$$

$$Y_a(s) = Y(s) + a_y/\,[s(s^2+\omega^2)]$$

$$Z_a(s) = Z(s) + a_z/\,[s(s^2+\omega^2)] - 2\omega\,a_x\,/\,[s^2(s^2+\omega^2)]$$

where $\{X(s), Y(s), Z(s)\}$ represent shorthand for the no-acceleration case terms. To get a final solution, we will need a few more inverse Laplace Transforms:

$$L^{-1}\{\,1/s^n\,\} = t^{n-1}\,/\,(n-1)!$$
$$L^{-1}\{\,1/[s^3(s^2+\omega^2)]\,\} = (1/\omega^4)\,[\,(1/2)\omega^2t^2 - \cos\omega t + 1\,]$$

So, taking the inverse Laplace Transform term-by-term yields:

$$x_a(t) = x(t) + (1/2)\, a_x t^2 + (2a_z/\omega^2)(\omega t - \sin \omega t) - (4a_x/\omega^2)[(1/2)\omega^2 t^2 - \cos \omega t + 1]$$

$$y_a(t) = y(t) + (a_y/\omega^2)(1 - \cos \omega t)$$

$$z_a(t) = z(t) + (a_z/\omega^2)(1 - \cos \omega t) - (2a_x/\omega^2)(\omega t - \sin \omega t)$$

which upon simplification and rearrangement becomes

$$x_a(t) = x(t) + (3/2)\, a_x t^2 + [\, 2a_z / \omega \,]\, t + [\, 4a_x / \omega^2 \,] - [\, 4a_x / \omega^2 \,] \cos \omega t - [\, 2a_z / \omega^2 \,] \sin \omega t$$

$$y_a(t) = y(t) + [\, a_y / \omega^2 \,] - [\, a_y / \omega^2 \,] \cos \omega t$$

$$z_a(t) = z(t) - [\, 2a_x / \omega \,]\, t + [\, a_z / \omega^2 \,] - [\, a_z / \omega^2 \,] \cos \omega t + [\, 2a_x / \omega^2 \,] \sin \omega t$$

Taking the derivative of the above yields the velocity equations:

$$x'_a(t) = x'(t) + 3a_x t + [\, 2a_z / \omega \,] + [\, 4a_x / \omega \,] \sin \omega t - [\, 2a_z / \omega \,] \cos \omega t$$

$$y'_a(t) = y'(t) + [\, a_y / \omega \,] \sin \omega t$$

$$z'_a(t) = z'(t) - [\, 2a_x / \omega \,] + [\, a_z / \omega \,] \sin \omega t + [\, 2a_x / \omega \,] \cos \omega t$$

These are the solutions to the constant-acceleration case of the CW Equations, as presented at the beginning of this report.

C. Generalized Acceleration

The generalized acceleration case of the CW Equations cannot be solved analytically. The material below is presented to illustrate this point.

The general CW Equations are

$$a_x(t) = (d^2x/dt^2) - 2\omega\,(dz/dt)$$
$$a_y(t) = (d^2y/dt^2) + \omega^2 y$$
$$a_z(t) = (d^2z/dt^2) + 2\omega\,(dx/dt) - 3\omega^2 z$$

where $\{a_x(t), a_y(t), a_z(t)\}$ are the generalized accelerations as a function of time. As before, we shall use Laplace Transforms to investigate this system of differential equations.

Taking the Laplace Transform of the above yields the following:

$$A_x(s) = [\, s^2 X_g(s) - s\, x(0+) - x'(0+) \,] - 2\omega [\, s\, Z_g(s) - z(0+) \,]$$
$$A_y(s) = [\, s^2 Y_g(s) - s\, y(0+) - y'(0+) \,] - \omega^2 Y_g(s)$$
$$A_z(s) = [\, s^2 Z_g(s) - s\, z(0+) - z'(0+) \,] + 2\omega [\, s\, X_g(s) - x(0+) \,] - 3\omega^2 Z_g(s)$$

where $\{A_x(s), A_y(s), A_z(s)\}$ are the Laplace Transforms of the generalized accelerations. Here, we use the subscript "g" to distinguish the appropriate Laplace Transform functions from the no-acceleration case.

Once again, we are left with a set of simultaneous equations to solve. Solving the above for $\{X_g(s), Y_g(s), Z_g(s)\}$ yields

$$X_g(s) = X(s) + A_x(s)/s^2 + 2\omega A_z(s)/[s(s^2+\omega^2)] - 4\omega^2 A_x(s)/[s^2(s^2+\omega^2)]$$

$$Y_g(s) = Y(s) + A_y(s)/(s^2+\omega^2)$$

$$Z_g(s) = Z(s) + A_z(s)/(s^2+\omega^2) - 2\omega\, A_x(s)/[s(s^2+\omega^2)]$$

where $\{X(s), Y(s), Z(s)\}$ represent shorthand for the no-acceleration case terms. To continue further is of limited use – the inverse Laplace Transforms of the terms with $\{A_x(s), A_y(s), A_z(s)\}$ result in convolution integrals which we cannot evaluate in general. Therefore, there exists no solution to the general CW Equations in the presence of generalized accelerations.