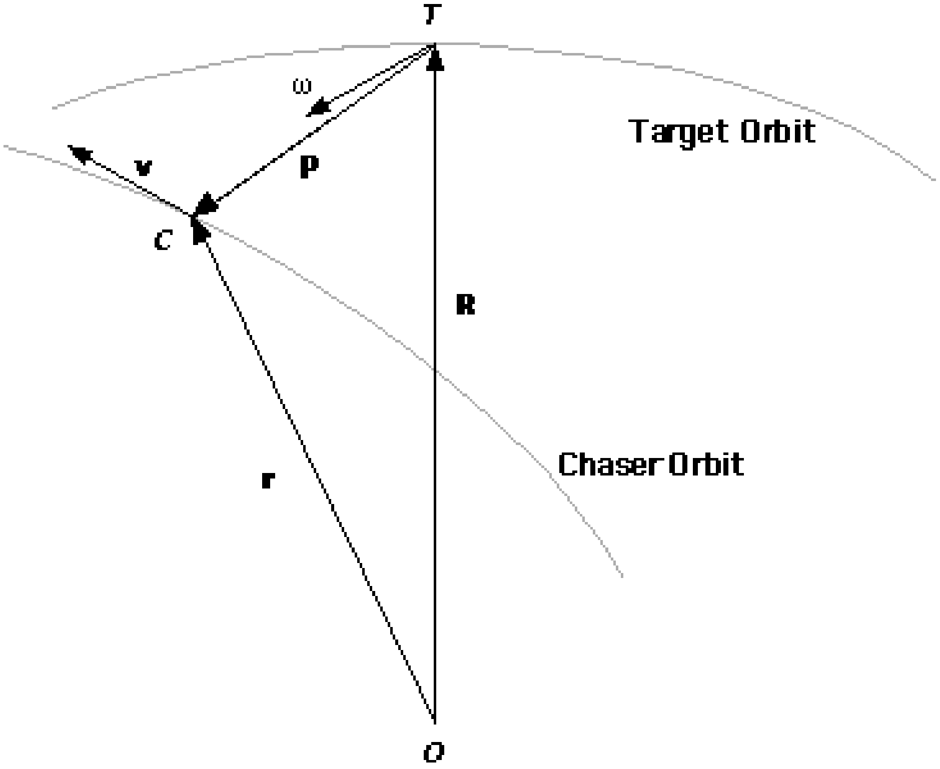
Derivation of the CW Equations

Consider the positions of two close orbiting objects called the ***target*** and ***chaser***. In an inertial frame, we can specify vectors **R** and **r** describing their positions.



Our goal is to write equations of motion of the chaser relative to the target. The popular approach (see Pearson's notes) uses a straight Newtonian formulation. Here, we shall use a Lagrangian formulation and Hamilton's principle.

(Aside: A Lagrangian formulation has the advantage of expressing the dynamics of a given system in terms of scalar quantities:

$$L = T - U$$

where L represents the Lagrangian,
T represents the kinetic energy of the system, and
U represents the potential energy of the system.

Hamilton's principle leads to the Lagrange equations of motion:

$$(d/dt)\{ \partial L / \partial(dq_i/dt) \} - (\partial L / \partial q_i) = f_i$$

where q_i represents the generalized coordinates of the system,

$\partial L / \partial q_i$ represents the partial derivatives of the Lagrangian with respect to the generalized coordinates of the system,

$\partial L / \partial(dq_i/dt)$ represents the partial derivatives of the Lagrangian with respect to the generalized momenta of the system, and

f_i represents the undetermined multipliers associated with the specific generalized coordinates.

Overall, the computations in a Lagrangian formulation are simpler than those in a Newtonian formulation, the latter of which makes use of vector forces and accelerations, and various complex vector operations.)

The Lagrangian of the chaser is defined by the difference between its kinetic and potential energies:

$$L = (1/2)\, m\, \mathbf{v}^2 - U(\mathbf{r})$$

Transforming the velocity to a rotating frame of reference yields

$$L = (1/2)\, m\, [\mathbf{v}_{rot} + (\omega \times \mathbf{r})]^2 - U(\mathbf{r})$$

which upon expansion is

$$L = (1/2)\, m\, \mathbf{v}^2_{rot} + m\, \mathbf{v}_{rot}\ (\omega \times \mathbf{r}) + (1/2)\, m\, (\omega \times \mathbf{r})^2 - U(\mathbf{r})$$

Now we shift the origin and express the position and velocity of the chaser relative to the target. Thus, we define

$$\mathbf{r} = \mathbf{R} + \mathbf{p}$$

and

$$\begin{aligned} \mathbf{v}_{rot} &= d\mathbf{r}/dt \\ &= d\mathbf{R}/dt + d\mathbf{p}/dt \\ &= d\mathbf{p}/dt \end{aligned}$$

since we notice that $d\mathbf{R}/dt = 0$ in the target-centered frame. Substituting these into the Lagrangian yields

$$L = (1/2)\, m(d\mathbf{p}/dt)^2 + m(d\mathbf{p}/dt)\ [(\omega \times (\mathbf{R}+\mathbf{p})] + (1/2)\, m\, [(\omega \times (\mathbf{R}+\mathbf{p})]^2 - U(\mathbf{R},\mathbf{p})$$

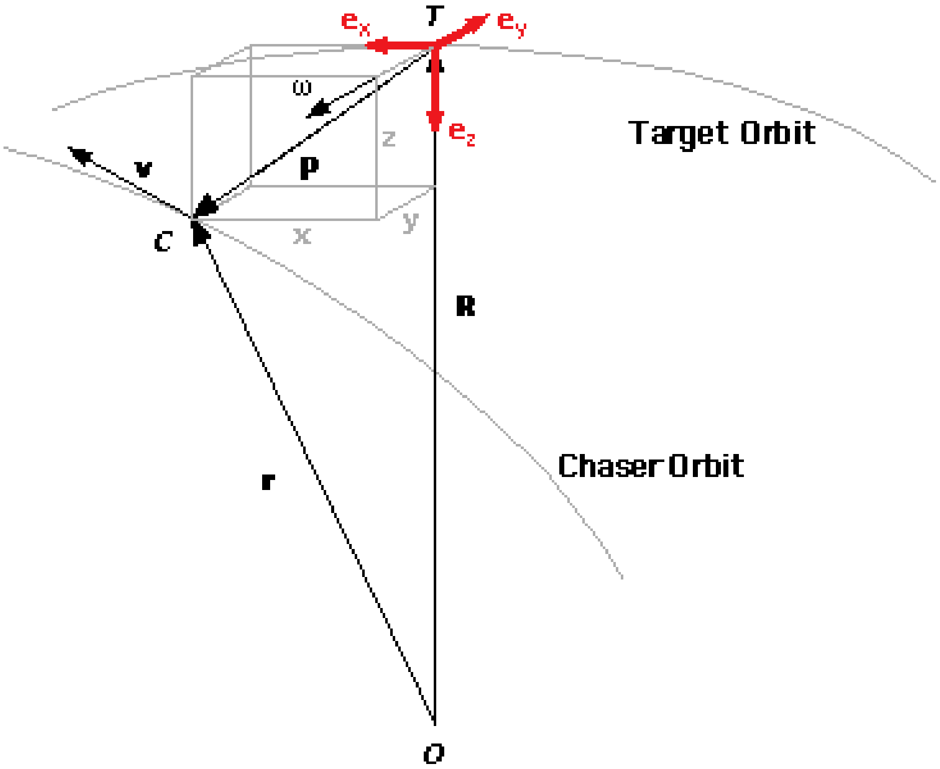
This is the Lagrangian for our target-centered system, describing the dynamics of the chaser relative to the target.

At this point, we will select a coordinate system to continue with the expansion. We shall consider two choices: Cartesian and Cylindrical-Polar.

Cartesian Formulation

We shall define three mutually orthogonal Cartesian axes described by the unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ such that

$$\mathbf{e}_z = -\mathbf{R} / ||\mathbf{R}||$$
$$\mathbf{e}_y = [(d\mathbf{p}/dt) \times \mathbf{R}] / ||\, [(d\mathbf{p}/dt) \times \mathbf{R}]\, ||$$
$$\mathbf{e}_x = \mathbf{e}_y \times \mathbf{e}_z$$



In our rotating frame of reference, the directions of the above axes are fixed with respect to time. This simplifies the process of taking derivatives with respect to time. We can express the vectors contained in the Lagrangian as thus:

$$\mathbf{p} = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z$$
$$\mathbf{R} = -R\,\mathbf{e}_z$$
$$\omega = -\omega\,\mathbf{e}_y$$
$$d\mathbf{p}/dt = (dx/dt)\,\mathbf{e}_x + (dy/dt)\,\mathbf{e}_y + (dz/dt)\,\mathbf{e}_z$$
$$\begin{aligned}\omega \times (\mathbf{R} + \mathbf{p}) &= (\omega \times \mathbf{R}) + (\omega \times \mathbf{p}) \\ &= (\omega R\,\mathbf{e}_x) + (-\omega z\,\mathbf{e}_x + \omega x\,\mathbf{e}_z) \\ &= -\omega\,(z - R)\,\mathbf{e}_x + \omega x\,\mathbf{e}_z\end{aligned}$$

Substituting the above into the Lagrangian and evaluating the various dot products yields

$$L = (1/2)\, m\, [(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2\,] + m\omega\, [x\, (dz/dt) - (z - R)\, (dx/dt)\,] + (1/2)\, m\omega^2\, [\, x^2 + (z - R)^2\,] - U\, (\mathbf{R},\mathbf{p})$$

Now, it is time to tackle the potential energy term. The potential energy of the chaser is due to gravity, which is solely a function of its position:

$$U\, (\mathbf{R},\mathbf{p}) = -G\, M\, m \,/\, \|\, \mathbf{R} + \mathbf{p}\, \|$$

Evaluating the norm of the vector **R** + **p** in Cartesian coordinates, and substituting the relationship $G\, M = \omega^2\, R^3$ into the above, yields

$$U\, (\mathbf{R},\mathbf{p}) = -m\, \omega^2\, R^3\, [\, x^2 + y^2 + (z - R)^2\,]^{-1/2}$$

Therefore, the Lagrangian expressed in Cartesian coordinates is

$$L = (1/2)\, m\, [(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2\,] + m\omega\, [x\, (dz/dt) - (z - R)\, (dx/dt)\,] + (1/2)\, m\omega^2\, [\, x^2 + (z - R)^2\,] + m\, \omega^2\, R^3\, [\, x^2 + y^2 + (z - R)^2\,]^{-1/2}$$

Now that we have the completely-expanded Lagrangian in Cartesian coordinates, we are ready to tackle the creation of the equations of motion. In Cartesian coordinates, the generalized coordinates q_i are simply {x, y, z}. We therefore will generate three equations:

$$(d/dt)\, \{\, \partial L \,/\, \partial(dx/dt)\, \} - (\partial L \,/\, \partial x) = f_x$$
$$(d/dt)\, \{\, \partial L \,/\, \partial(dy/dt)\, \} - (\partial L \,/\, \partial y) = f_y$$
$$(d/dt)\, \{\, \partial L \,/\, \partial(dz/dt)\, \} - (\partial L \,/\, \partial z) = f_z$$

For the x-equation,

$$\partial L \,/\, \partial x = m\omega\, (dz/dt) + m\, \omega^2\, x - m\, \omega^2\, R^3\, x\, [\, x^2 + y^2 + (z - R)^2\,]^{-3/2}$$

and

$$\begin{aligned}(d/dt)\, \{\, \partial L \,/\, \partial(dx/dt)\, \} &= (d/dt)\, \{\, m\, (dx/dt) - m\, \omega^2\, (z - R)\, \} \\ &= m\, (d^2x/dt^2) - m\, \omega\, (dz/dt)\end{aligned}$$

Therefore, the x equation of motion is

$$f_x = m\, (d^2x/dt^2) - m\omega\, (dz/dt) - m\omega\, (dz/dt) - m\omega^2\, x + m\omega^2\, R^3 x\, [x^2 + y^2 + (z-R)^2\,]^{-3/2}$$

which upon a little further reduction and rearranging is

$$f_x / m = (d^2x/dt^2) - 2\omega\,(dz/dt) - \omega^2\, x + \omega^2\, R^3x\, [x^2+y^2+(z-R)^2\,]^{-3/2}$$

Going through similar computations for y and z yields

$$f_y / m = (d^2y/dt^2) + \omega^2\, R^3y\, [x^2+y^2+(z-R)^2\,]^{-3/2}$$

and

$$f_z / m = (d^2z/dt^2) + 2\omega\,(dx/dt) - \omega^2\,(z - R) + \omega^2\, R^3z\, [x^2+y^2+(z-R)^2\,]^{-3/2}$$

The three equations of motion in Cartesian coordinates are thus:

$$a_x = (d^2x/dt^2) - 2\omega\,(dz/dt) - \omega^2\, x + \omega^2\, R^3x\, [x^2+y^2+(z-R)^2\,]^{-3/2}$$
$$a_y = (d^2y/dt^2) + \omega^2\, R^3y\, [x^2+y^2+(z-R)^2\,]^{-3/2}$$
$$a_z = (d^2z/dt^2) + 2\omega\,(dx/dt) - \omega^2\,(z-R) + \omega^2\, R^3z\, [x^2+y^2+(z-R)^2\,]^{-3/2}$$

where the new undetermined multipliers $\{a_x, a_y, a_z\}$ represent the accelerations of the system.

Notice that all three equations have the term $R^3\, [x^2+y^2+(z-R)^2\,]^{-3/2}$. If we make the assumption that $z << R$ and $(x^2 + y^2 + z^2) << R^2$, we can use the binomial expansion formula for small arguments:

$$(1 + \alpha)^{\pm n} = 1 \pm n\alpha \quad \text{if } \alpha << 1$$

Thus,

$$R^3\, [x^2+y^2+(z-R)^2\,]^{-3/2} = 1 + 3z/R - (3/2)\,(x^2+y^2+z^2)/\,R^2$$

If we substitute the above binomial expansion into our three equations of motion, we get

$$a_x = (d^2x/dt^2) - 2\omega\,(dz/dt) - \omega^2\, x + \omega^2\, x\, [1 + 3z/R - (3/2)\,(x^2+y^2+z^2)/\,R^2]$$
$$a_y = (d^2y/dt^2) + \omega^2\, y\, [1 + 3z/R - (3/2)\,(x^2+y^2+z^2)/\,R^2]$$
$$a_z = (d^2z/dt^2) + 2\omega\,(dx/dt) - \omega^2\,(z-R) + \omega^2\,(z-R)\, [1 + 3z/R - (3/2)\,(x^2+y^2+z^2)/\,R^2]$$

Simplifying a little further and dropping all terms on the order of $(1/R)^n$, where $n \geq 1$, results in the following equations:

$$a_x = (d^2x/dt^2) - 2\omega\,(dz/dt)$$
$$a_y = (d^2y/dt^2) + \omega^2\, y$$
$$a_z = (d^2z/dt^2) + 2\omega\,(dx/dt) - 3\omega^2\, z$$

Voila! These are the CW equations expressed in Cartesian coordinates.

Cylindrical-Polar Formulation

From the results of the Cartesian approach, we can see that the out-of-plane (Y) component of the CW equations is decoupled from the other two. One might be tempted to re-approach the solutions to the CW Equations with this thought in mind, and derive a cylindrical-polar formulation from first principles. We shall do just that in this section.

Let's start with the same Lagrangian as before:

$$L = (1/2)\, m(d\mathbf{p}/dt)^2 + m(d\mathbf{p}/dt)\ [(\omega \times (\mathbf{R}+\mathbf{p})] + (1/2)\, m\, [(\omega \times (\mathbf{R}+\mathbf{p})]^2 - U(\mathbf{R},\mathbf{p})$$

only this time we shall define <u>two</u> sets of three mutually orthogonal axes described by the unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, $\mathbf{e}_\eta$ and $\mathbf{e}_{\rho'}$, $\mathbf{e}_{\theta'}$, $\mathbf{e}_\eta$ such that

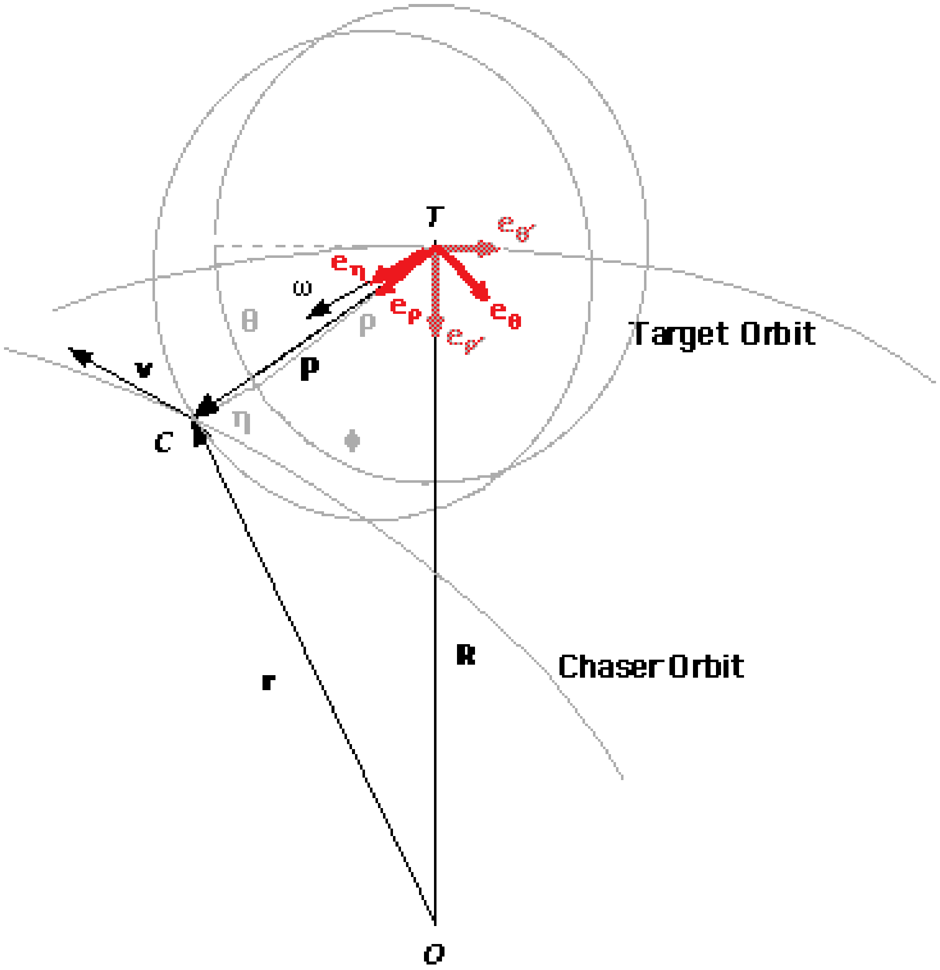
$\mathbf{e}_\rho$ points along the projection of p in the r-q plane;
$\mathbf{e}_{\rho'}$ points in the opposite direction of R;
$\mathbf{e}_h$ points in the angular momentum direction (i.e., same direction as w);
$\mathbf{e}_\theta$ completes the right-handed coordinate system of $\mathbf{e}_\rho$ and $\mathbf{e}_h$; and
$\mathbf{e}_{\theta'}$ completes the right-handed coordinate system of $\mathbf{e}_{\rho'}$ and $\mathbf{e}_h$

The primed set of axes are related to the unprimed by the expressions

$$\mathbf{e}_{\theta'} = \mathbf{e}_\theta \sin\theta - \mathbf{e}_\rho \cos\theta$$
$$\mathbf{e}_{\rho'} = \mathbf{e}_\theta \cos\theta + \mathbf{e}_\rho \sin\theta$$

In our rotating frame of reference, the directions of the ρ and θ axes change with respect to time. This complicates taking derivatives somewhat, but not too terribly so. We can define our vectors as thus:

$$\mathbf{p} = \rho\, \mathbf{e}_\rho + \eta\, \mathbf{e}_\eta$$
$$\mathbf{R} = -R\, \mathbf{e}_{\rho'}$$
$$\omega = \omega\, \mathbf{e}_\eta$$
$$d\mathbf{p}/dt = (d\rho/dt)\, \mathbf{e}_\rho + \rho\, (d\theta/dt)\, \mathbf{e}_\theta + (d\eta/dt)\, \mathbf{e}_\eta$$
$$\begin{aligned} \omega \times (\mathbf{R} + \mathbf{p}) &= (\omega \times \mathbf{R}) + (\omega \times \mathbf{p}) \\ &= (\omega R \cos\theta\, \mathbf{e}_\rho - \omega R \sin\theta\, \mathbf{e}_\theta) + (\omega\rho\, \mathbf{e}_\theta) \\ &= \omega R \cos\theta\, \mathbf{e}_\rho + \omega\, (\rho - R \sin\theta)\, \mathbf{e}_\theta \end{aligned}$$



Substituting the above relationships into the Lagrangian and evaluating the various dot products yields

$$
\begin{aligned}
L \;=\;& (1/2)\, m \left[(d\rho/dt)^2 + \rho^2 (d\theta/dt)^2 + (d\eta/dt)^2 \right] + \\
& m\omega \left[(d\rho/dt)\, R \cos\theta + \rho^2 (d\theta/dt) - \rho R \sin\theta\, (d\theta/dt) \right] + \\
& (1/2)\, m\omega^2 \left[R^2 \cos^2\theta + \rho^2 - 2\rho R \sin\theta + R^2 \sin^2\theta \right] - U(\mathbf{R},\mathbf{p})
\end{aligned}
$$

As in the Cartesian expansion, we have carried the potential energy term as $U(\mathbf{R},\mathbf{p})$. Our next task is to expand it, which we could do using brute force as we did before. However, the cylindrical symmetry of the problem gives us an elegant method of expanding the potential energy. We will get to that in a minute.

First, suppose that the out-of-plane component is small in a rectilinear sense. This means that the potential energy at point C (the location of the chaser) is the same as that at the point C_0 (i.e., the point with out-of-plane component = 0).

The vector pointing to C_0, in our Cylindrical-Polar frame, is $\rho\, \mathbf{e}_\rho$ (i.e., it is nothing more than the projection of **p** in-plane, or equivalently, **p** with $\eta = 0$.)

So, using the equivalence

$$U\,(\mathbf{R},\mathbf{p}) = U\,(\mathbf{R},\, \rho\, \mathbf{e}_\rho)$$

leads to

$$U\,(\mathbf{R},\mathbf{p}) \;=\; -G\,M\,m \,/\, \|\, \mathbf{R} + \rho\, \mathbf{e}_\rho \,\|$$

(Aside: Now, for the elegance mentioned earlier. The inverse distance between two points x_1 and x_0 can be expanded as follows:

$$1 \,/\, \|\, \mathbf{x}_1 - \mathbf{x}_0 \,\| \;=\; \sum_{\lambda=0}^{\infty} \left[\, (r_<)^\lambda \,/\, (r_>)^{\lambda+1} \,\right] P_\lambda\,(\cos \varphi)$$

where $r_<$ = length of the shorter vector, $r_>$ = length of the longer vector, and $P_\lambda\,(\cos \varphi)$ represent the Legendre polynomials as a function of the cosine of the inner angle between the two vectors.)

In our case, we can rewrite the denominator of the potential energy function as follows:

$$\begin{aligned} 1 \,/\, \|\, \mathbf{R} + \rho\, \mathbf{e}_\rho \,\| \;&=\; -1 \,/\, \|\, -\mathbf{R} - \rho\, \mathbf{e}_\rho \,\| \\ &=\; -\sum_{\lambda=0}^{\infty} \left[\, \rho^\lambda \,/\, R^{\lambda+1} \,\right] P_\lambda\,(\cos \varphi) \end{aligned}$$

since the length of the $-\mathbf{R}$ vector is still R, and the length of the vector $\rho\, \mathbf{e}_\rho$ is ρ, ρ is shorter than R, and P_λ represent the Legendre Polynomials as a function of the inner angle between the vectors $\rho\, \mathbf{e}_\rho$ and **R**.

After substituting $\omega^2 R^3$ for G M and $\sin \theta$ for $\cos \varphi$, we obtain the following expansion for the potential energy function:

$$U\,(\mathbf{R},\mathbf{p}) \;=\; -m\omega^2 \sum_{\lambda=0}^{\infty} \left[\, \rho^\lambda \,/\, R^{\lambda+1} \,\right] P_\lambda\,(\sin \theta)$$

leading to the cylindrical polar Lagrangian:

$$
\begin{aligned}
L \;=\;& (1/2)\, m\, [(d\rho/dt)^2 + \rho^2 (d\theta/dt)^2 + (d\eta/dt)^2] \;+ \\
& m\omega\, [(d\rho/dt)\, R \cos\theta \;+\; \rho^2 (d\theta/dt) \;-\; \rho R \sin\theta\, (d\theta/dt)] \;+ \\
& (1/2)\, m\omega^2\, [R^2 \cos^2\theta \;+\; \rho^2 \;-\; 2\rho R \sin\theta \;+\; R^2 \sin^2\theta] \;+ \\
& m\omega^2 \sum_{\lambda=0}^{\infty} [\rho^{\lambda} / R^{\lambda+1}]\, P_{\lambda} (\sin\theta)
\end{aligned}
$$

From here, we apply the Lagrangian to the equations of motion, one equation for each coordinate of the system.

The 3 equations of motion in Cylindrical-Polar coordinates are

$$
\begin{aligned}
(d/dt)\, \{ \partial L / \partial(d\rho/dt) \} \;-\; (\partial L / \partial\rho) \;&=\; f_{\rho} \\
(d/dt)\, \{ \partial L / \partial(d\theta/dt) \} \;-\; (\partial L / \partial\theta) \;&=\; f_{\theta} \\
(d/dt)\, \{ \partial L / \partial(d\eta/dt) \} \;-\; (\partial L / \partial\eta) \;&=\; f_{\eta}
\end{aligned}
$$

First, let’s evaluate the partial derivatives of the potential energy function, then proceed with the rest.

$$
\begin{aligned}
\partial U / \partial\rho \;&=\; -m\omega^2 \sum_{\lambda=1}^{\infty} [\lambda\rho^{\lambda-1} / R^{\lambda-2}]\, P_{\lambda}(\sin\theta) \\
\partial U / \partial\theta \;&=\; -m\omega^2 \sum_{\lambda=1}^{\infty} [\rho^{\lambda} / R^{\lambda-2}]\, [dP_{\lambda}(\sin\theta) / d(\sin\theta)] \cos\theta \\
\partial U / \partial\eta \;&=\; 0
\end{aligned}
$$

and the partial derivatives of the potential energy function with respect to the time-rate-of-change variables are all equal to 0. Now, using our assumption that $\rho << R$, we drop all terms of $\lambda=3$ or higher (i.e., all terms with R in the denominator), keeping only the $\lambda=1$ and $\lambda=2$ terms. If we use the following table of Legendre Polynomials:

l	$P_l(z)$	$dP_l(z) / dz$
0	1	0
1	z	1
2	$(1/2)(3z^2 - 1)$	$3z$

we obtain the following for the non-zero partial derivatives of the potential energy function:

$$
\begin{aligned}
\partial U / \partial\rho \;&=\; -m\omega^2\, [R \sin\theta \;+\; \rho\, (3 \sin^2\theta - 1)] \\
\partial U / \partial\theta \;&=\; -m\omega^2\, [r R \cos\theta \;+\; 3\rho^2 \cos\theta \sin\theta]
\end{aligned}
$$

Quite elegant, huh? Continuing with the partial derivatives of the rest of the Lagrangian (not shown here) and redefining in terms of new undetermined multipliers $a_\theta = f_\theta / m$ results in the following equations of motion:

$$a_\rho = (d^2\rho/dt^2) - \rho\,(d\theta/dt)^2 - 2\omega\,\rho\,(d\theta/dt) - 3\omega^2\rho\,\sin^2\theta$$
$$a_\theta = \rho\,(d^2\theta/dt^2) + 2\,(d\rho/dt)(d\theta/dt) + 2\omega\,(d\rho/dt) - 3\omega^2\rho\,\cos\theta\,\sin\theta$$
$$a_\eta = (d^2\eta/dt^2) + \omega^2\eta$$

These are the CW Equations expressed in Cylindrical-Polar form, seen earlier. One can demonstrate that these are identical to the Cartesian form by direct substitution.