CW Equations Case Studies and Rules of Thumb

The following is a series of case studies of the Clohessy–Wiltsire (CW) equations, describing relative motion in a frame defined by one of the objects. Derivations of the CW Equations is given at the end of this report. For completeness, the equations are stated here, in Cartesian and Cylindrical–Polar. The symmetries of the problem often dictate which formulation is the easiest to use, or is the most insightful.

A. Cartesian

$$a_x(t) = (d^2x/dt^2) - 2\omega\,(dz/dt)$$
$$a_y(t) = (d^2y/dt^2) + \omega^2 y$$
$$a_z(t) = (d^2z/dt^2) + 2\omega\,(dx/dt) - 3\omega^2 z$$

A.1. No–acceleration solution, with initial position {x(0+), y(0+), z(0+)} and initial velocity {x'(0+), y'(0+), z'(0+)}:

$$x(t) = [\,2z'(0+) / \omega + x(0+)\,] + [\,6\,\omega\,z(0+) - 3x'(0+)\,]\,t - [\,2z'(0+) / \omega\,] \cos \omega t + [\,4x'(0+) / \omega - 6z(0+)\,] \sin \omega t$$

$$y(t) = y(0+) \cos \omega t + [\,y'(0+) / \omega\,] \sin \omega t$$

$$z(t) = [\,4z(0+) - 2x'(0+) / \omega\,] + [\,2x'(0+) / \omega - 3z(0+)\,] \cos \omega t + [\,z'(0+) / \omega\,] \sin \omega t$$

and derivatives

$$x'(t) = [\,6\,\omega\,z(0+) - 3x'(0+)\,] + [\,2z'(0+)\,] \sin \omega t + [\,4x'(0+) - 6\omega z(0+)\,] \cos \omega t$$

$$y'(t) = -\omega y(0+) \sin \omega t + [\,y'(0+)\,] \cos \omega t$$

$$z'(t) = [\,3\omega z(0+) - 2x'(0+)\,] \sin \omega t + [\,z'(0+)\,] \cos \omega t$$

A.2. Constant–acceleration solution, with same initial position and velocity above, and constant acceleration $\{a_x, a_y, a_z\}$:

$$x_a(t) = x(t) + (3/2)\,a_x\,t^2 + [\,2a_z / \omega\,]\,t + [\,4a_x / \omega^2\,] - [\,4a_x / \omega^2\,] \cos \omega t - [\,2a_z / \omega^2\,] \sin \omega t$$

$$y_a(t) = y(t) + [\,a_y / \omega^2\,] - [\,a_y / \omega^2\,] \cos \omega t$$

$$z_a(t) = z(t) - [\,2a_x / \omega\,]\,t + [\,a_z / \omega^2\,] - [\,a_z / \omega^2\,]\cos\omega t + [\,2a_x / \omega^2\,]\sin\omega t$$

and derivatives

$$x'_a(t) = x'(t) + 3a_x t + [\,2a_z / \omega\,] + [\,4a_x / \omega\,]\sin\omega t - [\,2a_z / \omega\,]\cos\omega t$$

$$y'_a(t) = y'(t) + [\,a_y / \omega\,]\sin\omega t$$

$$z'_a(t) = z'(t) - [\,2a_x / \omega\,] + [\,a_z / \omega\,]\sin\omega t + [\,2a_x / \omega\,]\cos\omega t$$

B. Cylindrical–Polar

$$a_\rho(t) = (d^2\rho/dt^2) - \rho\,(d\theta/dt)^2 - 2\omega\,\rho\,(d\theta/dt) - 3\omega^2\rho\,\sin^2\theta$$
$$a_\theta(t) = \rho\,(d^2\theta/dt^2) + 2\,(d\rho/dt)(d\theta/dt) + 2\omega\,(d\rho/dt) - 3\omega^2\rho\,\cos\theta\,\sin\theta$$
$$a_\eta(t) = (d^2\eta/dt^2) + \omega^2\eta$$

No solutions are stated here *a priori*. Typically, one selects this formulation when one can state a constraint equation in terms of one of the cylindrical coordinates.

Free Trajectories

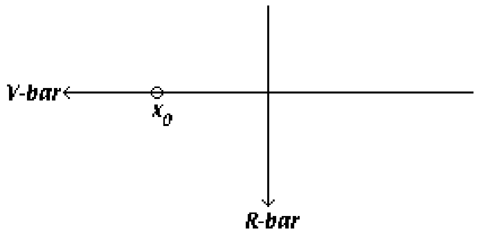
The first set of case studies examines the free evolution of the CW equations as a result of applying a select set of initial conditions. There are basically three such “free trajectories”: stationary, football, and cycloid.

Case Study 1. Sitting on the V-bar.

Here, we are sitting on the V-bar and are not applying any acceleration. So, if we consider the no-acceleration solution to the CW equations (Cartesian form), with initial condition $x(0+) = x_0$ and all other conditions = 0, we get the following:

$x(t) = x_0$
$y(t) = 0$
$z(t) = 0$

$x'(t) = 0$
$y'(t) = 0$
$z'(t) = 0$



Interpretation: If one is displaced along the V-bar by a distance x_0, then one will remain at that location as long as one does not apply an acceleration (a.k.a. Δv). This is what we call “sitting on the V-bar.”

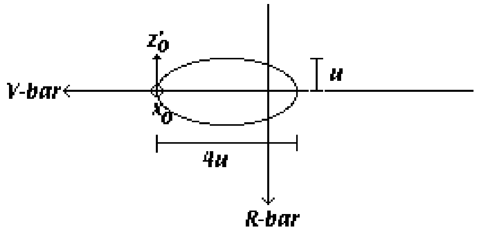
Rule of Thumb***: A chaser on the V-bar stays on the V-bar.***

Case Study 2. “Football”.

Consider the case of starting on the V-bar and having an initial velocity in the z-direction. If we consider the no-acceleration solution to the CW equations (Cartesian form), with initial condition $x(0+) = x_0$, $z'(0+) = z'_0$, and all other conditions = 0, we get the following:

$x(t) = x_0 + (2/\omega)\, z'_0\, (1 - \cos \omega t)$
$y(t) = 0$
$z(t) = (1/\omega)\, z'_0 \sin \omega t$

$x'(t) = 2\, z'_0 \sin \omega t$
$y'(t) = 0$
$z'(t) = z'_0 \cos \omega t$



Interpretation: From a simple examination of the equations, note the following:

$x(t)$ varies from 0 to $(4/\omega)\, z'_0$, offset by x_0;
$z(t)$ varies over $\pm(1/\omega)\, z'_0$;
the motions in $x(t)$ and $z(t)$ are exactly 90 degrees out of phase;
the maximum velocity in the x-direction is twice as large as the maximum in the z-direction.

The shape of the trajectory resembles that of a football, hence the name. (Actually, with the rounded ends, it more closely resembles a rugby ball than an American football.) The distance covered in the x-direction is 4 units, where 1 unit = $(1/\omega)\, z'_0$. The upward travel is 1 unit. Because there are no time-dependent terms other than the sinusoids, the chaser must return to the starting point after 1 period has elapsed, where 1 period = $2\pi/\omega$.

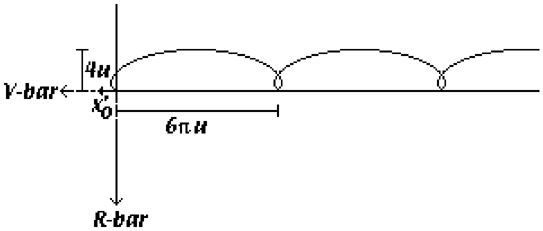
Rule of Thumb: ***Using a value of*** ω ***such that 1 period = 90 minutes,***
Δv_z ***= 1 fps up (down)   ~1000 ft up (down), ~4000 ft back (forward).***

Case Study 3. Posigrade and Retrograde Motion

Suppose that we start on the V-bar and have an initial velocity in the x-direction instead. If we consider the no-acceleration solution to the CW equations (Cartesian form), with initial condition $x'(0+) = x'_0$, and all other conditions = 0, we get the following:

$x(t) = -3x'_0\, t + (4/\omega)\, x'_0 \sin \omega t$
$y(t) = 0$
$z(t) = -(2/\omega)\, x'_0 (1 - \cos \omega t)$

$x'(t) = -3x'_0 + 4x'_0 \cos \omega t$
$y'(t) = 0$
$z'(t) = -2x'_0 \sin \omega t$



<u>Interpretation</u>: From an examination of the equations, note the following:

$x(t)$ is a sinusoid superimposed upon a linear term (a “prolate cycloid”);
$z(t)$ is a sinusoid which never changes sign.

If the initial velocity is applied to the left, the initial motion is to the left, but the predominant resultant motion is to the *right*. The time at which the motion changes from left to right (i.e., when $x'(t) = 0$) solves the equation

$$t_1 = (1/\omega) \text{ Arccos } (3/4)$$

Note that time t_1 is independent of the initial velocity x'_0. Furthermore, the chaser crosses the R-bar at a later time t_2 which is a solution to the equation

$$3/4 = (1 / \omega t_2) \sin \omega t_2$$

Once again, this time is independent of the initial velocity. Continuing to follow the chaser, we note that the detailed motion is that of a cycloid. The distance between re-arrivals on the V-bar is 6π units, and the change in height is 4 units. (Compare that to the football, which results in 4 units out and 1 unit in height.) The ratio between the V-bar re-arrivals and the change in height is about 5 to 1.

<u>Rules of Thumb</u>: ***Using a value of*** ω ***such that 1 period = 90 minutes,***

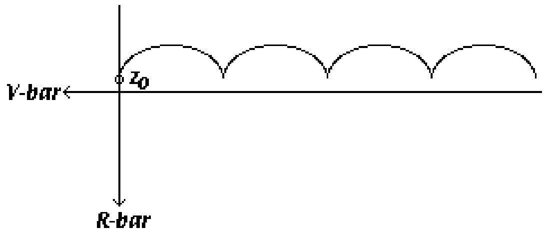
Δv_z = 1 fps left (right) ~4000 ft up (down), ~20000 ft back (forward).
For a posigrade burn, the chaser moves forward for ~10 minutes.
For a posigrade burn, the chaser crosses the negative R-bar at ~20 minutes.
Δv_z = 1 fps left (right) cross the R-bar at ~1000 ft above (below).

Case Study 4. “Can we sit passively on the R-bar?”

Suppose that we attempt to sit on the R-bar passively. If we consider the no-acceleration solution to the CW equations (Cartesian form), with initial condition $z(0+) = z_0$, and all other conditions = 0, we get the following:

$x(t) = 6z_0 (\omega t - \sin \omega t)$
$y(t) = 0$
$z(t) = z_0 + 3z_0 (1 - \cos \omega t)$

$x'(t) = 6\omega z_0 (1 - \cos \omega t)$
$y'(t) = 0$
$z'(t) = 3\omega z_0 \sin \omega t$



Interpretation: These equations are roughly of the same form as those giving rise to posigrade/retrograde motion (specifically in this case, a “curtate cycloid”). This illustrates that a chaser placed on the R-bar at a distance z_0 has the tendency to move lower and ahead if z_0 is positive, higher and behind if z_0 is negative. The velocity in the x-direction never changes sign, so the chaser will not return to the starting position (not counting if the chaser “laps” the target). The bottom line is *no*, we cannot sit passively on the R-bar.

OK, so let’s be a bit more proactive and attempt to stationkeep on the R-bar by applying constant accelerations. If we consider the constant-acceleration solution to the CW equations (Cartesian form), with accelerations a_x and a_z, initial condition $z(0+) = z_0$, and all other conditions = 0, we get the following set of equations:

$x_a(t) = -(3/2)\, a_x t^2 + (2/\omega)[3\omega^2 z_0 + a_z]\, t - (4/\omega^2)\, a_x \cos \omega t -$
$(2/\omega^2)[3\omega^2 z_0 + a_z] \sin \omega t + (4/\omega^2)\, a_x$
$y_a(t) = 0$
$z_a(t) = -(2/\omega)\, a_x t - (1/\omega^2)[3\omega^2 z_0 + a_z] \cos \omega t + (2/\omega^2)\, a_x \sin \omega t +$
$4z_0 + (1/\omega^2)\, a_x$
$x'_a(t) = -3a_x t + (2/\omega)[3\omega^2 z_0 + a_z] + (4/\omega)\, a_x \sin \omega t - (2/\omega)[3\omega^2 z_0 +$
$a_z] \cos \omega t$
$y'_a(t) = 0$
$z'_a(t) = -(2/\omega)\, a_x + (1/\omega)[3\omega^2 z_0 + a_z] \sin \omega t + (2/\omega)\, a_x \cos \omega t$

the solution to which is $a_x = 0$ and $a_z = -3\omega^2 z_0$.

Interpretation: The above solution dictates that one must thrust radially in an amount proportional to the offset distance and in a direction towards the target. One does not thrust laterally.

Rules of Thumb: ***Using a value of ω such that 1 period = 90 minutes,***

A chaser on the negative (positive) R-bar tends to drift posigrade (retrograde).

Vertical offset yields posigrade/retrograde burn of ~0.25 fps per 100 ft distance.

1200-ft vertical offset distance = ~3 fps posigrade/retrograde burn.

To stationkeep on the R-bar, the chaser must thrust towards the target.

Case Study 5. R-bar to V-bar Twice Orbit Rate Transition (TORVA)

Suppose we want to perform a transition from the R-bar to the V-bar at twice orbital rate such that we are not required to make any control inputs. Let's start with the no-acceleration solution to the x and z CW equations (Cartesian form):

$$x(t) = [\, 2z'(0+) / \omega + x(0+) \,] + [\, 6\,\omega\, z(0+) - 3x'(0+) \,]\, t - [\, 2z'(0+) / \omega \,] \cos \omega t + [\, 4x'(0+) / \omega - 6z(0+) \,] \sin \omega t$$

$$z(t) = [\, 4z(0+) - 2x'(0+) / \omega \,] + [\, 2x'(0+) / \omega - 3z(0+) \,] \cos \omega t + [\, z'(0+) / \omega \,] \sin \omega t$$

$$x'(t) = [\, 6\,\omega\, z(0+) - 3x'(0+) \,] + [\, 2z'(0+) \,] \sin \omega t + [\, 4x'(0+) - 6\omega z(0+) \,] \cos \omega t$$

$$z'(t) = [\, 3\omega z(0+) - 2x'(0+) \,] \sin \omega t + [\, z'(0+) \,] \cos \omega t$$

and work with the following initial conditions: $x(0+) = 0$, $z(0+) = z_0$, $x'(0+) = -2\omega t$, and $z'(0+) = z'_0$. Furthermore, let's stipulate that at V-bar arrival, $t_1 = 1/8$ (orbital period) $= \pi/4\omega$ and the conditions at that point are $x(t_1) = x_1$, $z(t_1) = 0$, $x'(t_1) = x'_1$, and $z'(t_1) = z'_1$. We would like to discover what we can about the relationships between the various constants.

The CW equation in z evaluated at t_1 is

$$0 = (1/2)\,(\sqrt{2})\, z_0 + (1/2\omega)\,(\sqrt{2})\, z'_0$$

which implies that $z'_0 = -\omega z_0$. This has an interesting consequence. If we plug in this relationship in the CW equation in x' evaluated at t_1, we get the result

$$x'_1 = 0$$

The velocity at V-bar arrival is found by evaluating the CW equation in z' at t_1

$$z'_1 = -(\sqrt{2})\, \omega z_0$$

<u>Interpretation</u>: To execute a TORVA requires that we meet a particular constraint on the initial closing velocity—the initial closing velocity should be directly proportional to the distance on the R-bar at which the TORVA is started. Furthermore, these conditions will lead to an arrival state on the V-bar which has no closing velocity.

Rules of Thumb: ***Using a value of*** ω ***such that 1 period = 90 minutes and*** z_0 ***= 600 ft,***

The initial transverse velocity should be 1.4 fps.
The initial closing velocity should be -0.70 fps.
V-bar arrival point is 500 ft with upward velocity = 1.0 fps and no closing velocity.

Constrained Trajectories

The next set of case studies examines the evolution of the CW equations in the presence of various constraint equations. These yield very interesting results, many of which have practical applications.

Case Study 6. Parallel V-bar Approach

In this case study we will investigate approaches parallel to the V-bar. To do this, consider the constant-acceleration solution to the CW equations (Cartesian form), with initial conditions $x(0+) = x_0$, $z(0+) = z_0$, $x'(0+) = x'_0$, and all other conditions = 0, subject to the following constraints:

$z_a(t) = z_0$ (no displacement along z-direction)
$z'_a(t) = 0$ (no velocity or motion in the z-direction)

If we simply plug in the initial conditions and constraints in the equation for $z_a(t)$ and rearrange the terms, we obtain

$$\begin{aligned} 0 = & -(2/\omega)\, a_x\, t + (1/\omega^2)\,[\,2\omega\, x'_0 - 3\omega^2 z_0 - a_z\,] \cos \omega t + \\ & (2/\omega^2)\, a_x \sin \omega t + (1/\omega^2)\,[\,a_z + 3\omega^2 z_0 - 2\omega\, x'_0\,] \end{aligned}$$

To satisfy this equation, the time-dependent terms must each equal zero. This leads to the result

$$a_x = 0$$
$$a_z = 2\omega\, x'_0 - 3\omega^2 z_0$$

Interpretation: To execute an approach parallel to the V-bar, we must thrust continually in the radial direction in an amount related to the offset from the V-bar and the initial closing velocity. Note that the displacement in the x-direction has no bearing on the amount of thrust required.

Substituting the above accelerations into the x-equations and simplifying leads to the following:

$$x_a(t) = x'_0\, t + x_0$$
$$x'_a(t) = x'_0$$

which shows that the closing velocity is constant.

Rules of Thumb:

To approach parallel to the V-bar, the chaser must thrust radially.
The closing velocity for a parallel V-bar approach is constant.

Case Study 7. R-bar Approach/Backaway.

Let's start with the x and z CW equations expressed in the Cartesian frame,

$$a_x(t) = (d^2x/dt^2) - 2\omega\,(dz/dt)$$
$$a_z(t) = (d^2z/dt^2) + 2\omega\,(dx/dt) - 3\omega^2 z$$

and constrain the motion to the R-bar. Let's not control the velocity along the R-bar, but simply observe what happens in that direction, freely. These requirements lead to the following constraint equations:

$x(t) = 0$ (no displacement along x-direction – thus, all derivatives are zero, too)
$a_z(t) = 0$ (no control inputs in z-direction)

If we simply apply the constraints to the CW equation in x, we obtain

$$a_x(t) = -2\omega\,(dz/dt)$$

Interpretation: This tells us that in order to constrain our motion to the R-bar, we must apply an acceleration in the x-direction proportional to our velocity in the z-direction; i.e., the faster we are travelling along the R-bar, the greater the acceleration required in the x-direction to maintain our motion along the R-bar.

With the same set of constraints, the CW equation in z simplifies to

$$0 = (d^2z/dt^2) - 3\omega^2 z$$

which has the general solution

$$z(t) = z_0 \cosh \alpha t + (z'_0 / \alpha) \sinh \alpha t$$

with derivative

$$z'(t) = -\alpha z_0 \sinh \alpha t + z'_0 \cosh \alpha t$$

where $\alpha = \omega\sqrt{3}$, and cosh and sinh are the hyperbolic cosine and sine functions, respectively.

Interpretation: There are several interesting conclusions to draw. First, if we start at the origin with a small initial velocity in the z-direction, our velocity will increase slowly at first, but eventually approach an exponential growth. Second, if one has an initial velocity towards the target, the velocity will decrease, stop, then increase the other way. The turnaround point occurs at a time t_1 obtained by setting the z'(t) equation above to zero, resulting in the solution

$$t_1 = (1/\alpha) \text{ Arctanh } (z'_0 / \alpha z_0)$$

Furthermore, the closest initial position z_0 which results in no collision at the turnaround point is obtained by plugging in the above result for time into the general solution and setting it to zero. This results in the relationship

$$z_0 = -(1/\alpha) z'_0$$

Rules of Thumb: ***Using a value of*** ω ***such that 1 period = 90 minutes (as needed),***

A chaser must thrust in the x-direction to constrain its motion to the R-bar.

A chaser on the R-bar will increase (decrease) its opening (closing) rate naturally.

The max closing velocity to avoid a collision is ~0.1 fps per 50 ft distance.

Case Study 8. Optimized Twice Orbit Rate Flyaround.

Let's start with the x and z CW equations expressed in the Cartesian frame,

$$a_x(t) = (d^2x/dt^2) - 2\omega\,(dz/dt)$$
$$a_z(t) = (d^2z/dt^2) + 2\omega\,(dx/dt) - 3\omega^2 z$$

and constrain the motion to a constant range circle with a constant transverse velocity such that we complete two circuits per orbital period in a clockwise direction. These requirements lead to the constraints

$$x'(t) = +2\omega\, z(t)$$
$$z'(t) = -2\omega\, x(t)$$

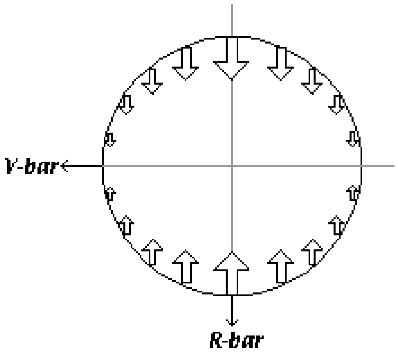
and by differentiation and re-substitution we obtain

$$x''(t) = -4\omega^2\, x(t)$$
$$z''(t) = -4\omega^2\, z(t)$$

If we apply the four above equations to the original CW equations and simplify, we get

$$a_x(t) = 0$$
$$a_z(t) = -3\omega^2 z$$

<ins>Interpretation</ins>: To fly a constant-range twice orbital rate flyaround, no control inputs are required in the x-direction. All control inputs are made in the z-direction; the magnitude is proportional to the distance from the V-bar and are directed towards the V-bar. The maximum control input is required at the R-bar crossings, and no control inputs are required at the V-bar crossings.



If our vehicle was fixed in attitude with respect to the Cartesian frame, this would be a simple guide to describe how to fly a TORF. However, we typically keep one vehicle axis pointed towards the center of the TORF and make inputs in the radial and transverse directions. This sets up ideally for investigation via the Cylindrical-Polar formulation of the CW equations. Let's start with those equations

$$a_\rho(t) = (d^2\rho/dt^2) - \rho\,(d\theta/dt)^2 - 2\omega\,\rho\,(d\theta/dt) - 3\omega^2\rho\,\sin^2\theta$$
$$a_\theta(t) = \rho\,(d^2\theta/dt^2) + 2\,(d\rho/dt)(d\theta/dt) + 2\omega\,(d\rho/dt) - 3\omega^2\rho\,\cos\theta\,\sin\theta$$

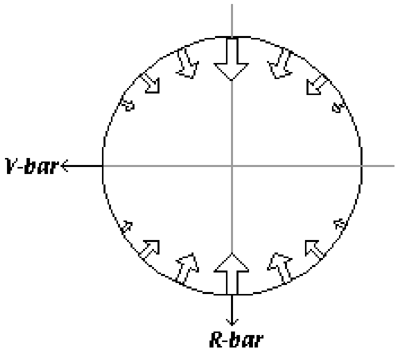
and constrain the motion as we did earlier. The constraints imply the following:

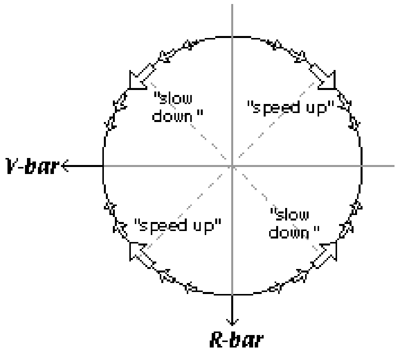
$$\rho(t) = R \text{ (where R is a constant)}$$
$$\theta'(t) = -2\omega$$

If we substitute these constraints into the Cylindrical-Polar CW equations (noting that the higher derivatives of the above are all equal to zero) and simplify, we obtain

$$a_\rho(t) = -3\omega^2\,R\,\sin^2\theta(t)$$
$$a_\theta(t) = -3\omega^2\,R\,\cos\theta(t)\,\sin\theta(t)$$

Interpretation: Since $\sin^2\theta$ is always ≥ 0, the radial control inputs are always negative or 0. Radial control inputs are at a maximum magnitude when $\sin^2\theta = 1$, and are zero when $\sin^2\theta = 0$. Transverse control inputs vary in sign depending on which quadrant of the flyaround the chaser is located; “speed-up” inputs are when $\cos\theta\sin\theta < 0$ (because of the leading negative sign), and “slow-down” inputs are required when $\cos\theta\sin\theta > 0$. Transverse control inputs are at a maximum magnitude when $\cos\theta = \sin\theta$, and are zero when $\cos\theta = 0$ or $\sin\theta = 0$.





Rules of Thumb:

To execute a TORF, the chaser must always direct radial thrust towards the center.
Radial thrust is at a maximum at R-bar crossings and zero at V-bar crossings.
Speed-up inputs are required in the lower left and upper right quadrants.
Slow-down inputs are required in the upper left and lower right quadrants.
No transverse inputs should be made near V-bar or R-bar crossings.